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# THE ZIQQURATH OF EXACT SEQUENCES OF $n$-GROUPOIDS

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RÉSUMÉ. Dans ce travail nous étudions la notion de suite exacte dans la sesqui-catégorie des $n$-groupoïdes. En utilisant les produits fibrés homotopiques, à partir d'un $n$-foncteur entre $n$-groupoïdes pointés nous construisons une suite de six ($n$-1)-groupoïdes. Nous montrons que cette suite est exacte dans un sens qui généralise les notions usuelles d'exactitude pour les groupes et les gr-catégories. En réitérant le processus, nous obtenons une ziggourat[1] de suites exactes de longueur croissante et dimension décroissante. Pour $n = 1$, nous retrouvons un résultat classic du à R. Brown et, pour $n = 2$, nous retrouvons ses généralisations dues à Hardie, Kamps et Kieboom et à Duskin, Kieboom et Vitale.

RÉSUMÉ. In this work we study exactness in the sesqui-category of $n$-groupoids. Using homotopy pullbacks, we construct a six term sequence of ($n$-1)-groupoids from an $n$-functor between pointed $n$-groupoids. We show that the sequence is exact in a suitable sense, which generalizes the usual notions of exactness for groups and categorical groups. Moreover, iterating the process, we get a ziqqurath[2] of exact sequences of increasing length and decreasing dimension. For $n = 1$, we recover a classical result due to R. Brown and, for $n = 2$, its generalizations due to Hardie, Kamps and Kieboom and to Duskin, Kieboom and Vitale.

## 1. Introduction

This work is a contribution to the general theory of higher dimensional categorical structures, like $n$-categories and $n$-groupoids. Examples and applications of higher dimensional categorical structures abound in mathematics and mathematical physics; the reader in search of good

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[1]Les Ziggourats (ou Ziggurats) étaient des temples en forme de pyramide à gradins répandus auprès des habitants de l'ancienne Mésopotamie [14].

[2]Ziqquraths (or Ziggurats) were a type of step pyramid temples common to the inhabitants of ancient Mesopotamia [14].

motivations can consult the books [5, 10, 11]. We focalize our attention on the study of homotopy pullbacks in the sesqui-category of $n$-groupoids and, more precisely, on the notion of exact sequence that can be expressed using homotopy pullbacks.
The problem we take as our guide-line is to generalize to $n$-groupoids a result established by R. Brown in the context of groupoids. Let us recall Brown's result from [1]: consider a fibration of groupoids

$$F\colon \mathbb{A} \to \mathbb{B}$$

and, for $a_0$ a fixed object of $\mathbb{A}$, consider the fibre $\mathcal{F}_{a_0}$ of $F$ over $a_0$. There is an exact sequence

$$\Pi_1(\mathcal{F}_{a_0}) \to \Pi_1(\mathbb{A}) \to \Pi_1(\mathbb{B}) \to \Pi_0(\mathcal{F}_{a_0}) \to \Pi_0(\mathbb{A}) \to \Pi_0(\mathbb{B})$$

where $\Pi_1(-)$ is the group of automorphisms of the base point and $\Pi_0(-)$ is the pointed set of isomorphism classes of objects.
The interest of Brown's result is that, despite its simplicity, it covers several quite different particular cases. We quote some of them:

1. A fibration $f\colon X \to Y$ of pointed topological spaces induces a fibration
$$F = \Pi_1(f)\colon \Pi_1(X) \to \Pi_1(Y)$$
on the homotopy groupoids; the sequence given by $F$ is the first part of the homotopy sequence of $f$.

2. For $G$ a fixed group, any extension $A \to B \to C$ of $G$-groups induces a fibration
$$F\colon \mathcal{Z}^1(G, B) \to \mathcal{Z}^1(G, C)$$
on the groupoids of derivations; the sequence given by $F$ is the fundamental exact sequence in non-abelian cohomology of groups.

3. Let $R$ be a commutative ring with unit, and consider

   $\mathbb{A}$ = the groupoid of Azumaya $R$-algebras and isomorpshisms of $R$-algebras.

$\mathbb{B}$ = the groupoid of Azumaya $R$-algebras and isomorphism classes of invertible bimodules.

As fibration $F$ we can consider the functor $F\colon \mathbb{A} \to \mathbb{B}$ which sends an isomorphism $f\colon A \to B$ to the invertible $A$-$B$-bimodule ${}_fB$, where the action of $A$ on ${}_fB$ is given by

$$A \otimes_R B \xrightarrow{f\otimes 1} B \otimes_R B \longrightarrow B$$

For $X$ a fixed Azumaya $R$-algebra, the sequence given by $F$ has the form

$$InnX \to AutX \to PicX \simeq PicR \to \pi_0(\mathcal{F}_X) \to \pi_0(\mathbb{A}) \to BrR$$

($Pic$ and $Br$ stay for Picard and Brauer, $Aut$ and $Inn$ are automorphisms and inner automorphisms of $R$-algebras). Such a sequence is an extension of the classical Rosenberg-Zelinsky exact sequence.

These examples suggest to look for an higher dimensional version of Brown's result. Indeed:

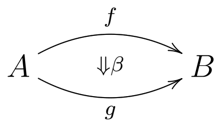
1. A fibration of pointed topological spaces also induces a morphism between the homotopy bigroupoids; a convenient generalization of Brown's result gives then the first 9 terms of the homotopy sequence (see [7] for more details).

2. Instead of an extension of $G$-groups, one can consider an extension of $\mathbb{G}$-crossed modules for $\mathbb{G}$ a fixed crossed module, or an extension of $\mathbb{G}$-categorical groups for $\mathbb{G}$ a fixed categorical group, and construct a morphism between the 2-groupoids of derivations; from such a morphism one can then obtain an exact sequence in non-symmetric cohomology of crossed modules or categorical groups (see [4] for more details).

3. The functor $F\colon \mathbb{A} \to \mathbb{B}$ of Example 3 can be easily modified so to have a morphism of bigroupoids:

   $\mathbb{B}$ = the bigroupoid of Azumaya $R$-algebras, invertible bimodules, and isomorphisms of bimodules.

$\mathbb{A}$ = the bigroupoid of Azumaya $R$-algebras, isomorphisms of $R$-algebras, and natural isomorphisms. A natural isomorphism




is an element $\beta$ of $B$ invertible with respect to the product and such that $\beta \cdot f(a) = g(a) \cdot \beta$ for all $a \in A$.

The morphism $F\colon \mathbb{A} \to \mathbb{B}$ is defined on a natural isomorphism $\beta$ by

$$F(\beta)\colon {}_fB \to {}_gB \quad F(\beta)(x) = \beta \cdot x.$$

(More in general, one can consider as $F$ the inclusion of enriched categories, equivalences and natural isomorphisms into enriched categories, invertible distributors and natural isomorphisms.)

With these examples in mind, we have developed the theory needed to state and prove our generalization of Brown's result: consider an $n$-functor $F\colon \mathbb{A} \to \mathbb{B}$ between pointed $n$-groupoids and its homotopy kernel $K\colon \mathbb{K} \to \mathbb{A}$

i- there is an exact sequence of ($n$-1)-pointed groupoids

$$\Pi_1(\mathbb{K}) \to \Pi_1(\mathbb{A}) \to \Pi_1(\mathbb{B}) \to \Pi_0(\mathbb{K}) \to \Pi_0(\mathbb{A}) \to \Pi_0(\mathbb{B})$$

ii- since $\Pi_0$ and $\Pi_1$ preserve exact sequences and commute each other, we can iterate the process and we get a ziqqurath of exact sequences

three pointed $n$-groupoids
six pointed ($n$-1)-groupoids
nine pointed ($n$-2)-groupoids
⋮
$3 \cdot n$ pointed groupoids
$3 \cdot (n+1)$ pointed sets

In particular, for $n = 1$ we obtain the two-level ziqqurath of Brown and, for $n = 2$, the three-level ziqqurath of [7] and [4].
The paper is organized as follows:

- In Section 2 we give the inductive (= enriched style) definition of $n$-Cat. We recall the definition of homotopy pullback and we prove that $n$-Cat is a sesqui-category with homotopy pullbacks.

- Section 3 is devoted to the definition of the sub-sesqui-category $n$-Gpd of $n$-groupoids, which is closed in $n$-Cat under homotopy pullbacks.

- In Section 4 we define exactness in the sesqui-category of pointed $n$-groupoids.

- The sesqui-functor $\Pi_0^{(n)} \colon n\text{-Gpd} \to (n\text{-}1)\text{-Gpd}$ is constructed in Section 5, and it is proved that it preserves exact sequences.

- Lax $n$-modifications are introduced in Section 6. We prove that homotopy pullbacks in $n$-Cat also satisfy a more sophisticated universal property expressed using lax $n$-modifications. This new universal property is needed in Sections 7, 8 and 9.

- In Section 7 we construct two sesqui-functors $\Pi_1^{(n)} \colon n\text{-Gpd}_* \to (n\text{-}1)\text{-Gpd}_*$ and $\Omega^{(n)} \colon n\text{-Gpd}_* \to n\text{-Gpd}_*$, and we prove that they preserve exact sequences. (Proposition 7.3 is proved using a result contained in the Appendix.)

- Finally, in Sections 8 and 9 we prove the main results: the fibration sequence and the ziqqurath of exact sequences associated with an $n$-functor between pointed $n$-groupoids.

Sections 2 and 6 are a survey of results from [13], they are inserted here for the reader's convenience. All along the paper, several proofs are omitted. Some of them are something more than a strightforward exercise. The interested reader can find all the details in [12].

## 2. The sesqui-category $n$-Cat

In this section we describe the sesqui-category $n$-Cat of strict $n$-categories, strict $n$-functors and lax $n$-transformations. We also describe homotopy pullbacks ($h$-pullbacks for short) in $n$-Cat. The definition of sesqui-category can be found in [15], for $h$-pullbacks see also [6].

2.1. Definition.

1. 0-Cat is the category of small sets and maps.

2. 1-Cat is the category of small categories and functors.

3. For $n > 1$, $n$-Cat has (strict) $n$-categories as objects and (strict) $n$-functors as morphisms.
   An $n$-category $\mathbb{C}$ consists of a set of objects $\mathbb{C}_0$, and for every pair $c_0, c_0' \in \mathbb{C}_0$, a ($n$-1)-category $\mathbb{C}_1(c_0, c_0')$. The structure is given by morphisms of ($n$-1)-categories:

$$\mathbb{I} \xrightarrow{u^0(c_0)} \mathbb{C}_1(c_0, c_0) \qquad \mathbb{C}_1(c_0, c_0') \times \mathbb{C}_1(c_0', c_0'') \xrightarrow{\circ^0_{c_0,c_0',c_0''}} \mathbb{C}_1(c_0, c_0'')$$

   called respectively 0-units and 0-compositions, with $c_0, c_0', c_0''$ any triple of objects of $\mathbb{C}$. Axioms are the usual ones for strict unit and strict associativity.
   An $n$-functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ consists of a map $F_0\colon \mathbb{C}_0 \to \mathbb{D}_0$ together with morphisms of ($n$-1)-categories

$$F_1^{c_0,c_0'}\colon \mathbb{C}_1(c_0, c_0') \to \mathbb{D}_1(F_0c_0, F_0c_0')$$

   with $c_0, c_0'$ any pair of objects of $\mathbb{C}$, such that usual strict functoriality axioms are satisfied.

2.2. Remark. The previous definition makes sense because one can prove by induction that $n$-Cat is a category with binary products and a terminal object $\mathbb{I}$.

2.3. Notation. Cell dimension will be often recalled as subscript: $c_k$ is a $k$-cell in the $n$-category $\mathbb{C}$. Moreover, if

$$c_k : c_{k-1} \to c'_{k-1} : c_{k-2} \to c'_{k-2} : \cdots \to \cdots : c_1 \to c'_1 : c_0 \to c'_0,$$

$c_k$ can be considered as an object of the ($n$-k)-category

$$\Big[ \cdots \big[ [\mathbb{C}_1(c_0, c'_0)]_1 (c_1, c'_1) \big]_1 \cdots \Big]_1 (c_{k-1}, c'_{k-1}).$$

In order to avoid this quite uncomfortable notation, the latter will be renamed more simply $\mathbb{C}_k(c_{k-1}, c'_{k-1})$, while $\mathbb{C}_k$ denotes the set of all $k$-cells in $\mathbb{C}$. Finally, 0-subscript of the underlying set of an $n$-category and 0-superscript of unit $u$ and composition $\circ$ will be often omitted.

2.4. Definition. Let $F, G \colon \mathbb{C} \to \mathbb{D}$ be morphisms of $n$-categories. By a 2-morphism $\alpha : F \Rightarrow G$ is meant:

1. The equality $F = G$ if $n = 0$.

2. A natural transformation $\alpha : F \Rightarrow G$ if $n = 1$.

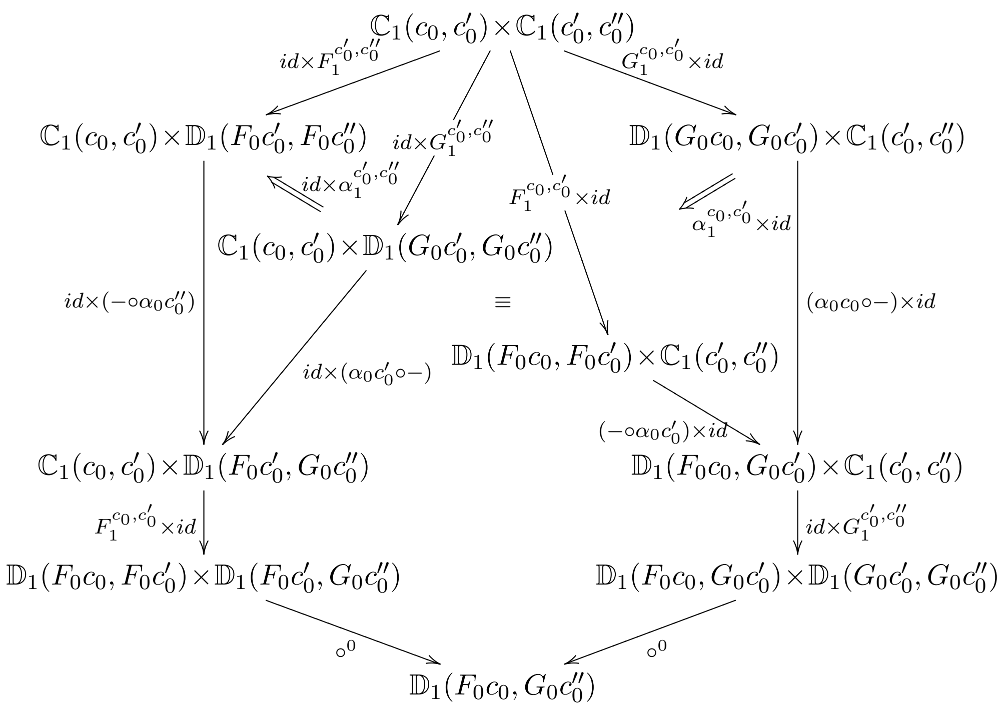
3. A lax $n$-transformation $\alpha : F \Rightarrow G$ if $n > 1$, that is, a pair $(\alpha_0, \alpha_1)$ where $\alpha_0 \colon \mathbb{C}_0 \to \mathbb{D}_1$ is a map such that $\alpha_0(c_0) = \alpha_{c_0} : Fc_0 \to Gc_0$, and $\alpha_1 = \{\alpha_1^{c_0, c'_0}\}_{c_0, c'_0 \in \mathbb{C}_0}$ is a collection of 2-morphisms of ($n$-1)-categories

$$\begin{array}{ccccc} & & \mathbb{C}_1(c_0, c'_0) & & \\ & \swarrow^{F_1^{c_0,c'_0}} & & \searrow^{G_1^{c_0,c'_0}} & \\ \mathbb{D}_1(F_0c_0, F_0c'_0) & & \overset{\alpha_1^{c_0,c'_0}}{\Longleftarrow} & & \mathbb{D}_1(G_0c_0, G_0c'_0) \\ & \searrow_{-\circ\alpha_0 c'_0} & & \swarrow_{\alpha_0 c_0 \circ -} & \\ & & \mathbb{D}_1(F_0c_0, G_0c'_0) & & \end{array} \qquad (1)$$

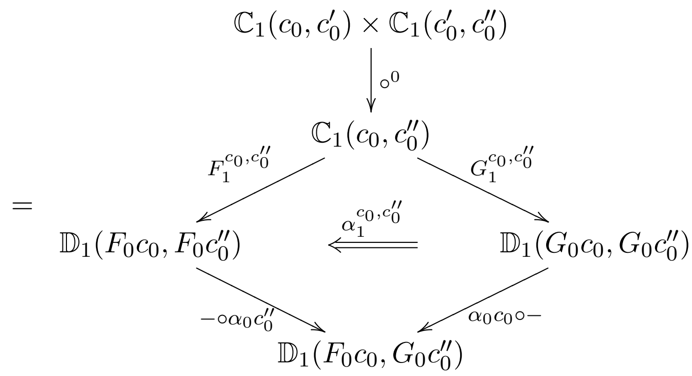
satisfying the following axioms:

– (*functoriality w.r.t. composition*) for every triple of objects

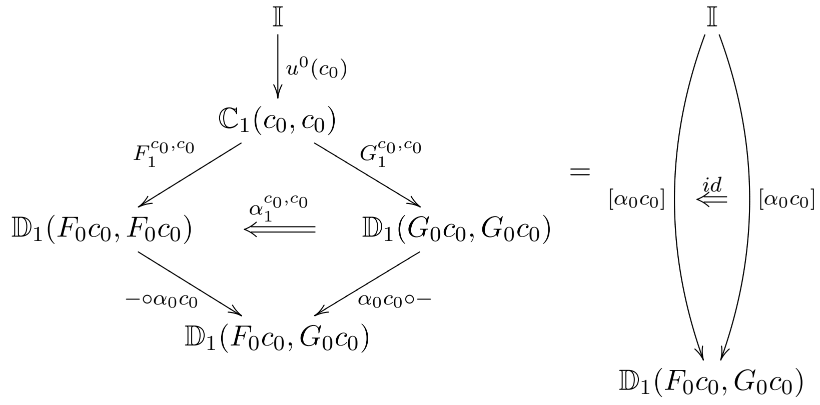
$c_0, c_0', c_0''$ of $\mathbb{C}_0$,




(2)



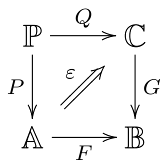
– (*functoriality w.r.t. units*) for every object $c_0$ of $\mathbb{C}_0$,



(3)

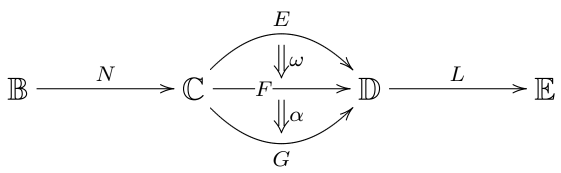
2.5. Proposition. *The category $n$-*Cat *equipped with lax $n$-transformations is a sesqui-category with $h$-pullbacks.*

Before proving the above proposition, let us recall the universal property of the $h$-pullback: consider two $n$-functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$. An $h$-pullback of $F$ and $G$ is a four-tuple $(\mathbb{P}, P, Q, \varepsilon)$




such that for any other four-tuple $(\mathbb{X}, M, N, \lambda : M \cdot F \Rightarrow N \cdot G)$ there exists a unique $L\colon \mathbb{X} \to \mathbb{P}$ satisfying $L \cdot P = M$, $L \cdot Q = N$, $L \cdot \varepsilon = \lambda$. This universal property defines $h$-pullbacks up to isomorphism.

Proof. We need vertical composition of lax $n$-transformations and reduced horizontal composition (whiskering). In fact, according to the following reference diagram

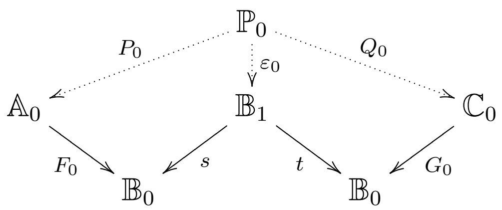


,

we let:

$$[\omega \cdot \alpha]_0(c_0) = \omega_0 c_0 \circ \alpha_0 c_0,$$
$$[\omega \cdot \alpha]_1^{c_0,c_0'} = \left(\alpha_1^{c_0,c_0'} \cdot (\omega_0 c_0 \circ -)\right) \cdot \left(\omega_1^{c_0,c_0'} \cdot (- \circ \alpha_0 c_0')\right);$$
$$[N \cdot \alpha]_0 = \alpha_0(N(b_0)), \qquad [N \cdot \alpha]_1^{b_0,b_0'} = N_1^{b_0,b_0'} \cdot \alpha_1^{Nb_0,Nb_0'};$$
$$[\alpha \cdot L]_0 = L(\alpha_0(c_0)), \qquad [\alpha \cdot L]_1^{c_0,c_0'} = \alpha_1^{c_0,c_0'} \cdot L_1^{Fc_0,Gc_0'}.$$

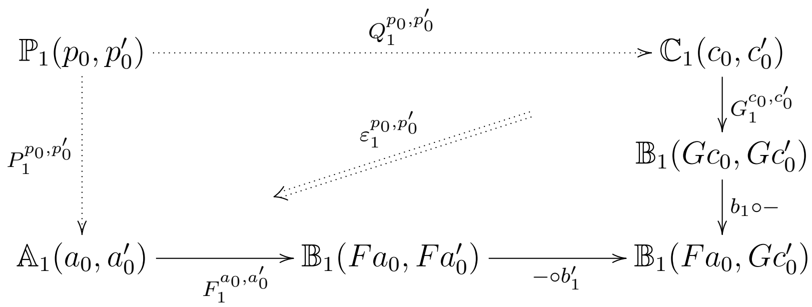
These constructions make $n$-Cat a sesqui-category.
An $h$-pullback in $n$-Cat can be described as follows.
For n=0, the usual pullback in Set is an instance of $h$-pullback, with the 2-morphism $\varepsilon$ being an identity.
For $n > 0$, we give an inductive construction of the standard $h$-pullback satisfying the universal property recalled above. The set $\mathbb{P}_0$ is the following limit in Set




where $s, t$ are *source* and *target* maps of 1-cells. More explicitly,

$$\mathbb{P}_0 = \{(a_0, b_1, c_0) \quad \text{s.t. } a_0 \in \mathbb{A}_0, c_0 \in \mathbb{C}_0, b_1 : Fa_0 \to Gc_0 \in \mathbb{B}_1\}$$
$$P_0((a_0, b_1, c_0)) = a_0, \quad Q_0((a_0, b_1, c_0)) = c_0, \quad \varepsilon_0((a_0, b_1, c_0)) = b_1$$

Let us fix two elements $p_0 = (a_0, b_1, c_0)$ and $p_0' = (a_0', b_1', c_0')$ of $\mathbb{P}_0$. The ($n$-1)-category $\mathbb{P}_1(p_0, p_0')$ is described by the following $h$-pullback in ($n$-1)-Cat:



The units and the compositions in $\mathbb{P}$ are determined by the universal property of the $h$-pullbacks $\mathbb{P}_1(p_0, p'_0)$. ∎

## 3. The sesqui-category $n$-Gpd

We first define equivalences of $n$-categories, and we use them to define $n$-groupoids.

3.1. Definition. An $n$-functor $F\colon \mathbb{C} \to \mathbb{D}$ is an equivalence if

(a) $F$ is essentially surjective on objects: for each object $d_0 \in \mathbb{D}_0$, there exist $c_0 \in \mathbb{C}_0$ and $d_1\colon Fc_0 \to d_0$ such that, for each $d'_0 \in \mathbb{D}_0$, the $(n$-$1)$-functors

$$d_1 \circ - \, : \mathbb{D}_1(d_0, d'_0) \to \mathbb{D}_1(Fc_0, d'_0)$$
$$- \circ d_1 \, : \mathbb{D}_1(d'_0, Fc_0) \to \mathbb{D}_1(d'_0, d_0)$$

are equivalences of $(n$-$1)$-categories, and

(b) for each $c_0, c'_0 \in \mathbb{C}_0$, the $(n$-$1)$-functor

$$F_1^{c_0,c'_0}\colon \mathbb{C}_1(c_0, c'_0) \to \mathbb{D}_1(Fc_0, Fc'_0)$$

is an equivalence of $(n$-$1)$-categories.

Essentially surjective $n$-functors and equivalences are closed under composition and stable under $h$-pullback.

3.2. Definition. A 1-cell $c_1\colon c_0 \to c'_0$ of an $n$-category $\mathbb{C}$ is an equivalence if, for each $c''_0 \in \mathbb{C}_0$, the $(n$-$1)$-functors

$$c_1 \circ - \, : \mathbb{C}_1(c'_0, c''_0) \to \mathbb{C}_1(c_0, c''_0) \qquad - \circ c_1 \, : \mathbb{C}_1(c''_0, c_0) \to \mathbb{C}_1(c''_0, c'_0)$$

are equivalences of $(n$-$1)$-categories.

3.3. Definition. An $n$-category $\mathbb{C}$ is an $n$-groupoid if

(a) every 1-cell of $\mathbb{C}$ is an equivalence, and

(b) for each $c_0, c'_0 \in \mathbb{C}_0$, the $(n$-$1)$-category $\mathbb{C}_1(c_0, c'_0)$ is an $(n$-$1)$-groupoid.

3.4. Remark. In the context of strict $n$-categories, the previous definition of $n$-groupoid is equivalent to those given by Street [15] and Kapranov and Voevodsky [9]. This fact is not easy to prove and a detailed proof can be found in [8]. In the same paper we also show that Definition 3.1 and Definition 3.2 are indeed redundant. In fact

- in Definition 3.1, $d_1 \circ -$ is an equivalence if, and only if, $- \circ d_1$ is;
- in Definition 3.2, $c_1 \circ -$ is an equivalence if, and only if, $- \circ c_1$ is.

We denote by $n$-Gpd the full sub-sesqui-category of $n$-Cat having as objects $n$-groupoids. The following result is straightforward.

3.5. Proposition. *The sesqui-category* $n$-Gpd *is closed in* $n$-Cat *under h-pullbacks.*

We denote by $n$-Gpd$_\star$ the sequi-category of pointed $n$-groupoids: a pointed $n$-groupoid is an $n$-groupoid $\mathbb{C}$ together with a fixed object $\star \in \mathbb{C}_0$, an $n$-functor $F\colon \mathbb{C} \to \mathbb{D}$ is pointed if $F(\star_{\mathbb{C}}) = \star_{\mathbb{D}}$, a lax $n$-transformation $\alpha\colon F \Rightarrow G$ is pointed if $\alpha(\star_{\mathbb{C}}) = u^0_{\star_{\mathbb{D}}}$. Once again, $h$-pullbacks in $n$-Gpd$_\star$ are constructed as in $n$-Cat.

## 4. Exact sequences

To define exactness, we need a notion of surjectivity suitable for $n$-categories.

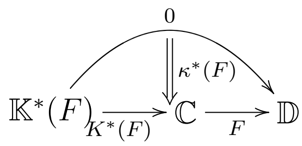
4.1. Definition. An $n$-functor $F\colon \mathbb{C} \to \mathbb{D}$ is $h$-surjective if

(a) it is essentially surjective on objects (see Definition 3.1), and

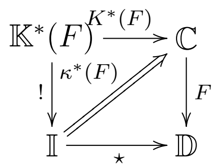
(b) for each $c_0, c_0' \in \mathbb{C}_0$, the ($n$-1)-functor $F_1^{c_0,c_0'}$ is $h$-surjective.

Once again, $h$-surjective functors are closed under composition and stable under $h$-pullbacks. Moreover, an $n$-functor is an equivalence iff it is $h$-surjective and faithful (i.e., injective on $n$-cells).

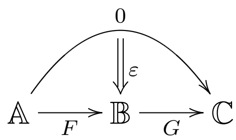
If $F\colon \mathbb{C} \to \mathbb{D}$ is an $n$-functor in $n$-Gpd$_\star$, we denote by



its $h$-kernel, that is the $h$-pullback



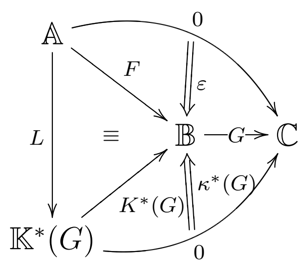
4.2. Definition. Let the following diagram in $n$-Gpd$_\star$ be given:



We call the triple $(F, \varepsilon, G)$ *exact in* $\mathbb{B}$ if the comparison $n$-functor

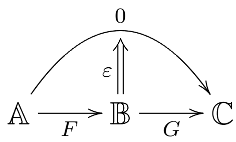
$$L\colon \mathbb{A} \to \mathbb{K}^*(G)$$

given by the universal property of the $h$-kernel $(\mathbb{K}^*(G), K^*(G), \kappa^*(G))$, is $h$-surjective.

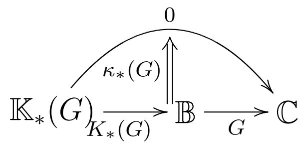


Observe that for $n = 0$ this is the usual definition of exact sequence of pointed sets, and for $n = 1$ this is the notion of 2-exactness introduced in [17] for categorical groups. In fact, for $n = 1$ $h$-surjective precisely means full and essentially surjective.

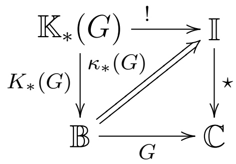
4.3. Remark. Analogously, we say that the triple $(F, \varepsilon, G)$



is exact in $\mathbb{B}$ if the comparison $n$-functor $L\colon \mathbb{A} \to \mathbb{K}_*(G)$ is $h$-surjective, where



is the $h$-pullback



## 5. The sesqui-functors $\Pi_0^{(n)}$ and $D^{(n)}$

In this section we define two sesqui-functors

$$n\text{-Gpd} \underset{D^{(n)}}{\overset{\Pi_0^{(n)}}{\rightleftarrows}} (n\text{-}1)\text{-Gpd}$$

(see [16] for the definition of sesqui-functor).

5.1. Definition. (The functor $\Pi_0^{(n)}$)

1. $\Pi_0^{(1)}$ is the functor Gpd $\to$ Set assigning to a groupoid $\mathbb{C}$ the set $|\mathbb{C}|$ of isomorphism classes of objects of $\mathbb{C}$.

2. For $n > 1$, let an $n$-groupoid $\mathbb{C}$ be given. Then
$$\Pi_0^{(n)}\mathbb{C} = ([\Pi_0^{(n)}\mathbb{C}]_0, [\Pi_0^{(n)}\mathbb{C}]_1(-,-))$$
where $[\Pi_0^{(n)}\mathbb{C}]_0 = \mathbb{C}_0$ and $[\Pi_0^{(n)}\mathbb{C}]_1(c_0, c_0') = \Pi_0^{(n-1)}(\mathbb{C}_1(c_0, c_0'))$. Compositions and units are obtained inductively: ${}^{\Pi_0^{(n)}\mathbb{C}}\circ = \Pi^{(n-1)}({}^{\mathbb{C}}\circ)$,

$u(c_0) = \Pi_0^{(n-1)}(u(c_0))$.
Now let an $n$-functor $F : \mathbb{C} \to \mathbb{D}$ be given. Then $[\Pi_0^{(n)} F]_0 = F_0$ and $[\Pi_0^{(n)} F]_1^{c_0,c_0'} = \Pi_0^{(n-1)}(F_1^{c_0,c_0'})$ define $\Pi_0^{(n)}$ on morphisms.

Note that the previous definition makes sense because one can prove inductively that $\Pi_0^{(n)}$ preserves binary products and the terminal object $\mathbb{I}$.

5.2. Definition. (The sesqui-functor $\Pi_0^{(n)}$)

1. Since $[\Pi_0^{(2)}\mathbb{D}]_1 = \mathbb{D}_1/\sim$, the quotient of $\mathbb{D}_1$ under the equivalence relation $\sim$ identifying 1-cells $d_1, d_1' \colon d_0 \to d_0'$ if there exists a 2-cell $d_2 \colon d_1 \to d_1'$, we define $[\Pi_0''\alpha]_0 = \alpha_0 \cdot p$, where $p$ is the canonic projection on the quotient.

2. For $n > 2$, let $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$ be a 2-morphism. We define $\Pi_0^{(n)}\alpha$ by $[\Pi_0^{(n)}\alpha]_0 = \alpha_0$ and $[\Pi_0^{(n)}\alpha]_1^{c_0,c_0'} = \Pi_0^{(n-1)}(\alpha_1^{c_0,c_0'})$.

A careful use of induction shows that $\Pi_0^{(n)}$ is well-defined and is indeed a sesqui-functor.

The definition of the sesqui-functor $D^{(n)}$ is easier. We make it explicit only on objects.

5.3. Definition. (The sesqui-functor $D^{(n)}$)

1. $D^{(1)}$ is the functor (= trivial sesqui-functor) $D^{(1)}$ : Set $\to$ Gpd assigning to a set $C$ the discrete groupoid $D(C)$ with objects the elements of $C$ and only identity arrows.

2. For $n > 1$, let an ($n$-1)-groupoid $\mathbb{C}$ be given. Then $D^{(n)}$ is given by $[D^{(n)}\mathbb{C}]_0 = \mathbb{C}_0$ and $[D^{(n)}\mathbb{C}]_1(c_0, c_0') = D^{(n-1)}(\mathbb{C}_1(c_0, c_0'))$.

It is an exercise for the reader to prove the following fact which, in particular, implies that $D^{(n)}$ preserves $h$-pullbacks.

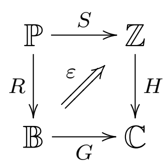
5.4. Proposition. *For all $n > 1$, there is an adjunction of sesqui-functors*

$$\Pi_0^{(n)} \dashv D^{(n)} ,$$

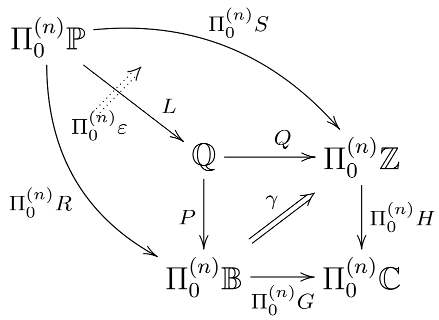
*and therefore an adjunction of the underlying functors.*

We are going to prove the main result of this section: the sesqui-functor $\Pi_0^{(n)}$ preserves exact sequences. Two preliminary lemmas clarify the relations between preservation of exactness and its main ingredients: $h$-surjectivity and the notion of $h$-pullback.

5.5. LEMMA. *Let us consider the following h-pullback diagram:*




*The comparison* $L : \Pi_0^{(n)}\mathbb{P} \to \mathbb{Q}$ *with the h-pullback of* $\Pi_0^{(n)}(G)$ *and* $\Pi_0^{(n)}(H)$ *is h-surjective.*




Proof. By induction on $n$.
1) For $n = 1$, the $h$-pullback $\mathbb{P}$ has objects and arrows

$$(b_0,\ Gb_0 \xrightarrow{c_1} Hz_0\ , z_0), \quad (b_1, =, z_1) : (b_0, c_1, z_0) \to (b'_0, c'_1, z'_0)$$

where the "=" stays for the commutative square $c_1 \circ Hz_1 = Gb_1 \circ c'_1$. Hence the set $\Pi_0^{(1)}(\mathbb{P})$ has elements the classes $[b_0, c_1, z_0]_\sim$. On the other side, the set $\mathbb{Q}$ is a usual pullback in Set. It has elements the pairs $([b_0]_\sim, [z_0]_\sim)$ such that $\Pi_0^{(1)}G([b_0]_\sim) = \Pi_0^{(1)}H([z_0]_\sim)$, *i.e.* $[Gb_0]_\sim = [Hz_0]_\sim$, *i.e.* such that there exists $c_1 : Gb_0 \to Hz_0$. Then the comparison $L = L_0 : [b_0, c_1, z_0]_\sim \mapsto ([b_0]_\sim, [z_0]_\sim)$ is clearly surjective.
2) For $n = 2$, the $h$-pullback $\mathbb{P}$ is a 2-groupoid with objects

$$(b_0,\ Gb_0 \xrightarrow{c_1} Hz_0\ , z_0).$$

Arrows and 2-cells are of the form

$$(b_1, c_2, z_1) : (b_0, c_1, z_0) \to (b'_0, c'_1, z'_0), \quad (b_2, =, z_2) : (b_1, c_2, z_1) \to (b'_1, c'_2, z'_1)$$

Therefore the groupoid $\Pi_0^{(2)}\mathbb{P}$ has objects $(b_0, c_1, z_0)$ and arrows $[b_1, c_2, z_1]_\sim$. On the other side, the groupoid $\mathbb{Q}$ has objects and arrows

$$(b_0,\ Gb_0 \xrightarrow{[c_1]_\sim} Hz_0\ , z_0), \quad ([b_1]_\sim, =, [z_1]_\sim)$$

with $[b_1]_\sim : b_0 \to b'_0$ in $\Pi_0^{(2)}\mathbb{B}$ and $[z_1]_\sim : z_0 \to z'_0$ in $\Pi_0^{(2)}\mathbb{Z}$ such that the diagram

$$\begin{array}{ccc} Gb_0 & \xrightarrow{[c_1]_\sim} & Hz_0 \\ {\scriptstyle [Gb_1]_\sim}\downarrow & & \downarrow{\scriptstyle [Hz_1]_\sim} \\ Gb'_0 & \xrightarrow[{[c'_1]_\sim}]{} & Hz'_0 \end{array} \tag{4}$$

commutes. Hence the comparison

$$\begin{aligned} L \quad : \quad & (b_0, c_1, z_0) \mapsto (b_0, c_1, z_0) \\ & [b_1, c_2, z_1]_\sim \mapsto ([b_1]_\sim, [z_1]_\sim) \end{aligned}$$

is $h$-surjective. In fact it is an identity on objects, and full on homs. Let us fix a pair of objects $(b_0, c_1, z_0)$ and $(b'_0, c'_1, z'_0)$ in the domain, and an arrow $([b_1]_\sim, =, [z_1]_\sim)$ in $\mathbb{Q}$, where the "=" express the commutativity of the diagram (4) above. Then $[c_1 \circ Hz_1]_\sim = [Gb_1 \circ c'_1]_\sim$ if, and only if, there exists

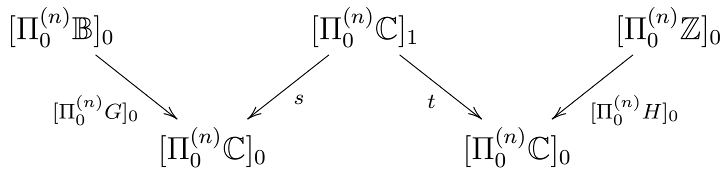
$$c_2 : c_1 \circ Hz_1 \to Gb_1 \circ c'_1.$$

In other words we get an arrow $[b_1, c_2, z_1]_\sim$ of $\Pi_0^{(2)}\mathbb{P}$ sent by $L$ to $([b_1]_\sim, = , [z_1]_\sim)$, *i.e.* $L$ is full.

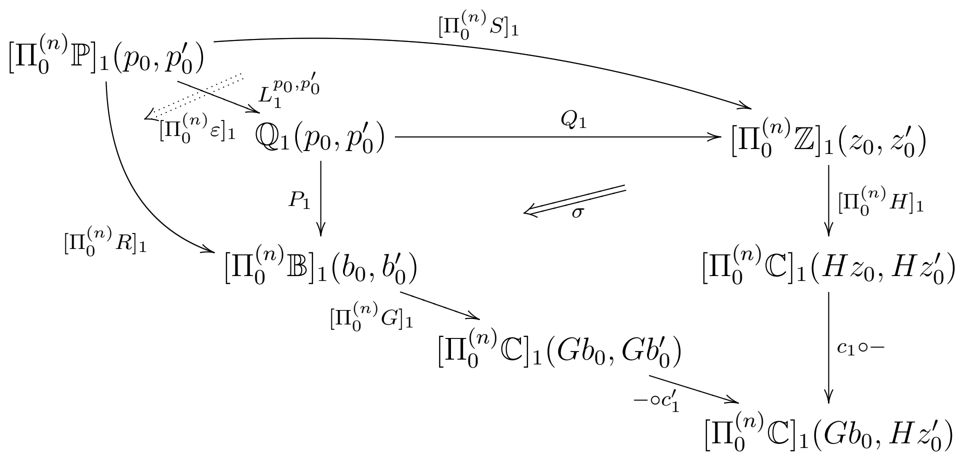
3) Finally, let $n > 2$. On objects, $L_0 : [\Pi_0^{(n)}\mathbb{P}]_0 \to \mathbb{Q}_0$ is the identity. In fact, $[\Pi_0^{(n)}\mathbb{P}]_0 = \mathbb{P}_0$, the set-theoretical limit over the diagram

$$\begin{array}{ccccccc} \mathbb{B}_0 & & & \mathbb{C}_1 & & & \mathbb{Z}_0 \\ & {\scriptstyle G_0}\searrow & \swarrow{\scriptstyle s} & & {\scriptstyle t}\searrow & \swarrow{\scriptstyle H_0} & \\ & & \mathbb{C}_0 & & \mathbb{C}_0 & & \end{array}$$

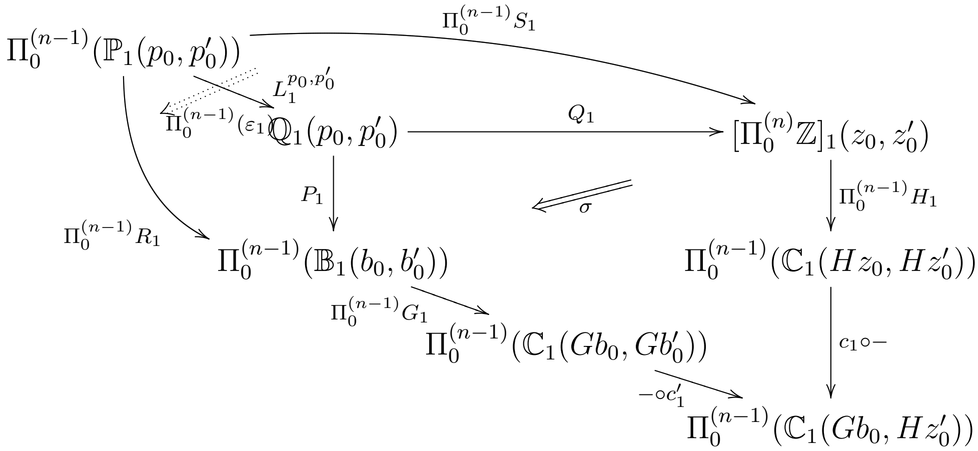
and, for $n > 2$, this diagram coincides with the one defining $\mathbb{Q}_0$:




On homs, let us fix two objects $p_1 = (b_0, c_1, z_0)$ and $p_1' = (b_0', c_1', z_0')$ of $[\Pi_0^{(n)}\mathbb{P}]_0 = \mathbb{P}_0$ and compute $L_1^{p_1,p_1'}$ by means of universal property of $h$-pullbacks. The diagram




is the same as (and determined by)



This shows that $L_1^{p_0,p_0'}$ is itself a comparison between $\Pi_0$ of an $h$-pullback and an $h$-pullback of a $\Pi_0$ of a diagram (of ($n$-1)-groupoids), hence it is $h$-surjective by induction hypothesis. ■

5.6. LEMMA. *If an $n$-functor $L : \mathbb{A} \to \mathbb{K}$ is $h$-surjective, then also $\Pi_0^{(n)}(L)$ is $h$-surjective.*

Proof. By induction on $n$.
1) For $n = 1$, let $L$ be an $h$-surjective functor between groupoids, *i.e.* $L$ is full and essentially surjective on objects. Therefore, for an element $[k_0]_\sim \in \Pi_0^{(1)}\mathbb{K}$ there exists a pair $(a_0, k_1 : La_0 \to k_0)$. Hence $(\Pi_0^{(1)}L)([a_0]_\sim) = [La_0]_\sim = [k_0]_\sim$.
2) For $n = 2$, let $L$ be an $h$-surjective morphism between 2-groupoids. Explicitely, this means that

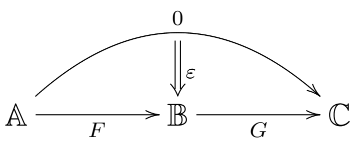
(i) for any $k_0$ there exist $(a_0, k_1 : La_0 \to k_0)$, and

(ii) for any pair $a_0, a_0'$, $L_1^{a_0,a_0'}$ is $h$-surjective.

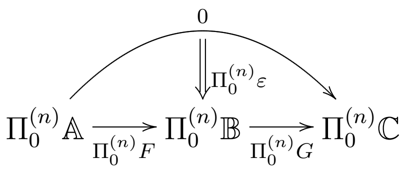
Since $[\Pi_0^{(2)}L]_0 = L_0$, for any $k_0$ one has $[k_1]_\sim : La_0 \to k_0$, and this proves the first condition on $\Pi_0^{(2)}L$. Moreover, once we fix a pair $a_0, a_0'$ of objects, by definition one has $[\Pi_0^{(2)}L]_1^{a_0,a_0'} = \Pi_0^{(1)}(L_1^{a_0,a_0'})$. Hence it is $h$-surjective by the previous case.
3) Finally, let $n > 2$. A morphism $L$ of $n$-groupoids is $h$-surjective when conditions (i) and (ii) above hold. Since $[\Pi_0^{(n)}L]_0 = L_0$ and $[\Pi_0^{(n)}(\mathbb{K})]_1 = \Pi_0^{(n-1)}(\mathbb{K}_1)$, condition (i) for $\Pi_0^{(n)}L$ precisely is condition (i) for $L$, hence it holds. Moreover, whence we fix a pair $a_0, a_0'$ of objects, by definition one has $[\Pi_0^{(n)}L]_1^{a_0,a_0'} = \Pi_0^{(n-1)}(L_1^{a_0,a_0'})$. Hence it is $h$-surjective by induction hypothesis. ■

We are ready to state and prove the main result of this section.

5.7. PROPOSITION. *Given an exact sequence in $n$-*Gpd$_*$

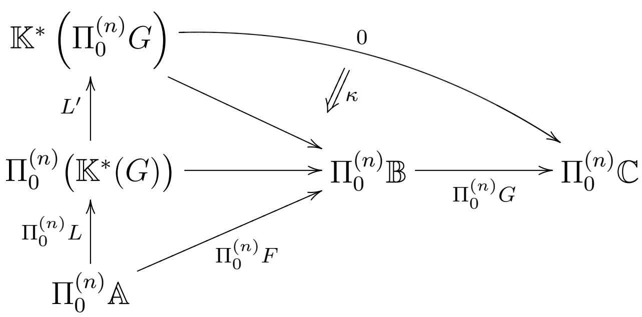


*the sequence*




*is exact in (n-1)*-Gpd$_*$.

Proof. Let us consider the diagram

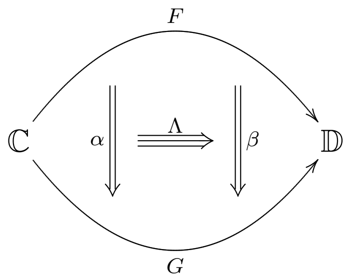



where $L$ is the comparison in $n$-Gpd (notations as in Definition 4.2). $L$ is $h$-surjective by hypothesis. Therefore $\Pi_0^{(n)}L$ is $h$-surjective by Lemma 5.6. Moreover, $L'$ is the comparison in ($n$-1)-Gpd, so that it is $h$-surjective by Lemma 5.5. Finally, their composition is again $h$-surjective, and it is the comparison between $\Pi_0^{(n)}\mathbb{A}$ and the kernel of $\Pi_0^{(n)}G$ by uniqueness in the universal property of $h$-kernels. ∎

## 6. Lax $n$-modifications

We already have two sesqui-functors $\Pi_0^{(n)}$ and $D^{(n)}$. In Section 7 we will construct two other sesqui-functors $\Pi_1^{(n)}$ and $\Omega^{(n)}$. In order to define $\Omega^{(n)}$ on lax $n$-transformations, we use the fact that $h$-pullbacks in $n$-Gpd$_\star$ satisfay another universal property. To express this new universal property we introduce here lax $n$-modifications between lax $n$-transformations.

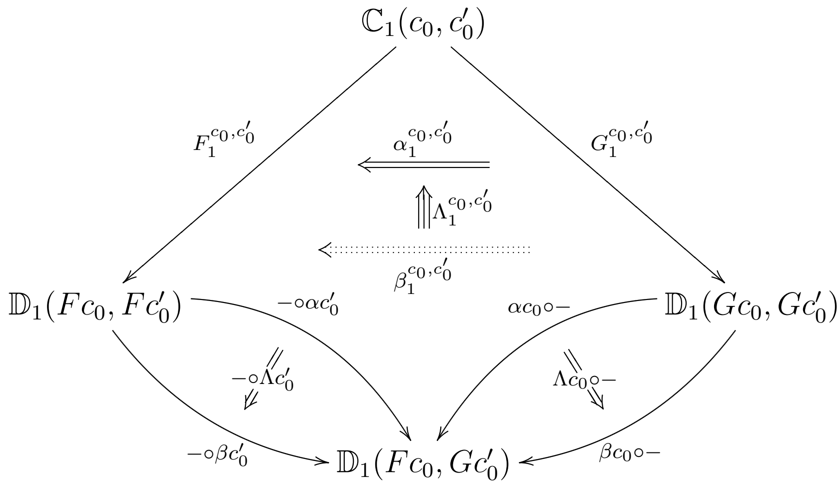
6.1. DEFINITION. Let $\alpha, \beta\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ be 2-morphisms of $n$-categories. By a 3-morphism $\Lambda : \alpha \Rrightarrow \beta$




is meant:

1. The equality $\alpha = \beta$ if $n = 1$.

2. A lax $n$-modification $\Lambda : \alpha \Rrightarrow \beta$ if $n > 1$, that is, a pair $\langle \Lambda_0, \Lambda_1 \rangle$, where $\Lambda_0\colon \mathbb{C}_0 \to \mathbb{D}_2$ is a map such that, for every $c_0$ in $\mathbb{C}_0$, $\Lambda_0(c_0)\colon \alpha_0(c_0) \to \beta_0(c_0)$, and $\Lambda_1 = \{\Lambda_1^{c_0,c_0'}\}_{c_0,c_0' \in \mathbb{C}_0}$ is a collection of 3-morphisms of ($n$-1)-categories that fill the following diagrams:



*i.e.*

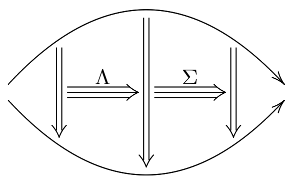
$$\begin{array}{ccc} G_1^{c_0,c_0'} \cdot (- \circ \alpha c_0') & \xrightarrow{\alpha_1^{c_0,c_0'}} & F_1^{c_0,c_0'} \cdot (- \circ \alpha c_0') \\ {\scriptstyle G_1^{c_0,c_0'}\cdot(\Lambda c_0 \circ -)} \Downarrow & \Lambda_1^{c_0,c_0'} \Rrightarrow & \Downarrow {\scriptstyle F_1^{c_0,c_0'}\cdot(-\circ\Lambda c_0)} \\ G_1^{c_0,c_0'} \cdot (\beta c_0 \circ -) & \xrightarrow[\beta_1^{c_0,c_0'}]{} & F_1^{c_0,c_0'}\cdot(-\circ \beta c_0') \end{array}$$

These data must obey to *functoriality* axioms described by equations in ($n$-1)-Cat, see [13].

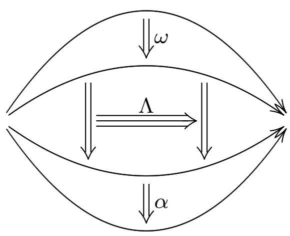
In the pointed case we ask moreover that $\Lambda_0(\star)$ is the identity 2-cell.

Once equipped with lax $n$-modifications, the sequi-categories $n$-Cat, $n$-Gpd and $n$-Gpd$_\star$ are sesqui$^2$-categories, see [13]. This essentially means that:

- there are 2-compositions of lax $n$-modifications as $\Lambda \cdot \Sigma$:

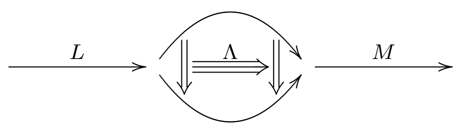



  and reduced 1-compositions of lax $n$-modifications as $\omega \cdot \Lambda$ and $\Lambda \cdot \alpha$:




  so that homs are sequi-categories;

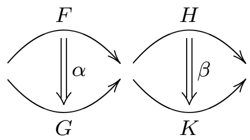
- there is reduced horizontal 0-composition of lax $n$-modifications as $L \cdot \Lambda$ and $\Lambda \cdot M$:



so that composing with an $n$-functor gives a sesqui-functor between homs;

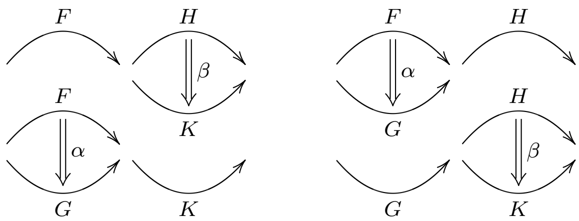
- there is horizontal 0-composition of lax $n$-transformations

$$\alpha * \beta \colon \alpha \setminus \beta \Longrightarrow \alpha/\beta \colon F \cdot H \Longrightarrow G \cdot K$$




where domain and codomain 2-morphisms are respectively

$$\alpha \setminus \beta := (F \cdot \beta) \cdot (\alpha \cdot K) \qquad (\alpha \cdot H) \cdot (G \cdot \beta) =: \alpha/\beta$$




(5)

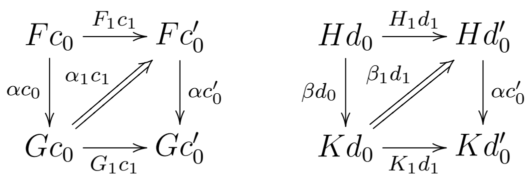
so that 0-composing with a 2-morphism gives a lax natural transformation of sesquifunctors.

The full definition of sesqui$^2$-category, with the remaining compatibility axioms, and the fact that $n$-Cat is a sesqui$^2$-category can be found in [12, 13], see also [2, 3] for the horizontal 0-composition of lax $n$-transformations. Here we just recall the inductive definition of $\alpha * \beta$:

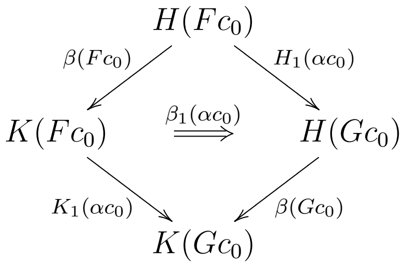
(a) for any $c_0 \in \mathbb{C}_0$, $[\alpha * \beta]_0(c_0) = \beta_1(\alpha_0(c_0))$

(b) for any $c_0, c_0' \in \mathbb{C}_0$, $[\alpha * \beta]_1^{c_0,c_0'} = \alpha_1^{c_0,c_0'} * \beta_1^{Fc_0,Gc_0'}$

6.2. Remark. For sake of clarity, let us write explicitly $\alpha * \beta$ for $n = 2$. Given $c_1 \colon c_0 \to c_0'$ in $\mathbb{C}$ and $d_1 \colon d_0 \to d_0'$ in $\mathbb{D}$, the lax 2-transformations $\alpha$ and $\beta$ are specified by

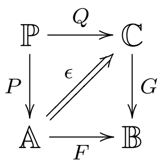


and the lax 2-modification $\alpha * \beta$ is given by

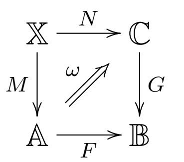


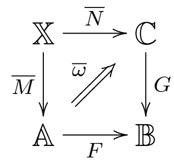

The following proposition holds in $n$-Cat as well as in $n$-Gpd and in $n$-Gpd$_\star$ (for $h^2$-pullbacks see also [6]).

6.3. PROPOSITION. *The h-pullback described in Section 2*

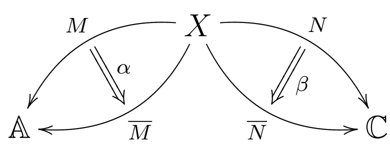



*satisfies the following universal property: for every pair of four-tuples* $(\mathbb{X}, M, N, \omega)$ *and* $(\mathbb{X}, \overline{M}, \overline{N}, \overline{\omega})$

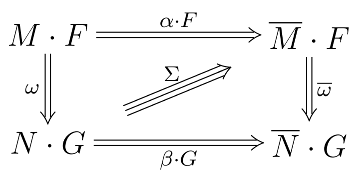



*for every pair of lax n-transformations* $\alpha$ *and* $\beta$




*and for every lax n-modification* $\Sigma$



*there exists a unique lax n-transformation* $\lambda : L \Rightarrow \overline{L} : \mathbb{X} \to \mathbb{P}$ *such that*

$$\lambda \cdot P = \alpha, \qquad \lambda \cdot Q = \beta, \qquad \lambda * \varepsilon = \Sigma.$$

*We will recall this property as the universal property of the* $h^2$*-pullback.*

Proof. The $n$-functors $L$ and $\overline{L}$ are given by the universal property of the $h$-pullback applied respectively to $(\mathbb{X}, M, N, \omega)$ and $(\overline{\mathbb{X}}, \overline{M}, \overline{N}, \overline{\omega})$. As far as $\lambda$ is concerned, one has:

- for any $x_0 \in \mathbb{X}_0$, $\lambda x_0 = (\alpha x_0, \Sigma x_0, \beta x_0) \colon Lx_0 \to \overline{L}x_0$
- for any $x_0, x_0' \in \mathbb{X}_0$, $\lambda_1^{x_0,x_0'}$ is given by the universal property of the $h^2$-pullback $\mathbb{P}_1(Lx_0, \overline{L}x_0')$ in $(n$-1)-Cat:

$$\lambda_1^{x_0,x_0'} \colon \overline{L}_1^{x_0,x_0'} \cdot (\lambda_0 x_0 \circ -) \Rightarrow L_1^{x_0,x_0'} \cdot (- \circ \lambda_0 x_0')$$

is the unique lax $(n$-1)-transformation such that

$$\lambda_1^{x_0,x_0'} \cdot Q_1^{Lx_0,\overline{L}x_0'} = \beta_1^{x_0,x_0'},\ \lambda_1^{x_0,x_0'} \cdot P_1^{Lx_0,\overline{L}x_0'} = \alpha_1^{x_0,x_0'},\ \lambda_1^{x_0,x_0'} * \epsilon_1^{Lx_0,\overline{L}x_0'} = \Sigma_1^{x_0,x_0'}$$

■

## 7. The sesqui-functors $\Pi_1^{(n)}$ and $\Omega^{(n)}$

We start with a first, easy description of the sesqui-functor

$$\Pi_1^{(n)} \colon n\text{-Gpd}_\star \to (n\text{-}1)\text{-Gpd}_\star$$

7.1. Definition. (The sesqui-functor $\Pi_1^{(n)}$)

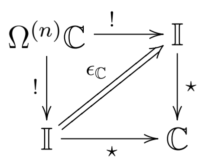
- Let $\mathbb{C}$ be a pointed $n$-groupoid. We define $\Pi_1^{(n)}\mathbb{C} = \mathbb{C}_1(\star, \star)$. It is a pointed $(n$-1)-groupoid with the identity 1-cell as base point.
- Let $F \colon \mathbb{C} \to \mathbb{D}$ be an $n$-functor in $n$-Gpd$_\star$. Since $F(\star) = \star$, we get an $(n$-1)-functor $\Pi_1^{(n)}F = F_1^{\star,\star} \colon \mathbb{C}_1(\star,\star) \to \mathbb{D}_1(\star,\star)$
- Let $\alpha \colon F \Rightarrow G \colon \mathbb{C} \to \mathbb{D}$ be a lax $n$-transformation in $n$-Gpd$_\star$. Since $\alpha_0(\star) = u(\star)$, we get a lax $(n$-1)-transformation $\Pi_1^{(n)}\alpha = \alpha_1^{\star,\star} \colon G_1^{\star,\star} \to F_1^{\star,\star}$.

It is easy to check that $\Pi_1^{(n)}\colon n\text{-Gpd}_\star \to (n\text{-}1)\text{-Gpd}_\star$ is a sesqui-functor contravariant on lax $n$-transformations.

Despite its simplicity, the previous definition is quite difficult to use in our setting, because it is not an inductive definition. For this reason, we look for a different decription of $\Pi_1^{(n)}$.

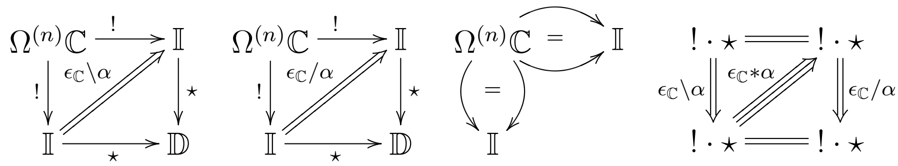
7.2. Definition. (The sesqui-functor $\Omega^{(n)}$)

- Let $\mathbb{C}$ be a pointed $n$-groupoid. We define $\Omega^{(n)}\mathbb{C}$ by the following $h$-pullback




- Let $F\colon \mathbb{C} \to \mathbb{D}$ be an $n$-functor in $n$-Gpd$_\star$. The universal property of the $h$-pullback $\Omega^{(n)}\mathbb{D}$ gives a unique $n$-functor $\Omega^{(n)}F\colon \Omega^{(n)}\mathbb{C} \to \Omega^{(n)}\mathbb{D}$ such that $\Omega^{(n)}F \cdot \epsilon_{\mathbb{D}} = \epsilon_{\mathbb{C}} \cdot F$.

- Let $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ be a lax $n$-transformation in $n$-Gpd$_\star$. Consider the following situation




The universal property of the $h^2$-pullback $\Omega^{(n)}\mathbb{D}$ gives a unique lax $n$-transformation

$$\Omega^{(n)}\alpha\colon \Omega^{(n)}G \Rightarrow \Omega^{(n)}F\colon \Omega^{(n)}\mathbb{C} \to \Omega^{(n)}\mathbb{D}.$$

such that $\Omega^{(n)}\alpha * \epsilon_{\mathbb{D}} = \epsilon_{\mathbb{C}} * \alpha$.

It is easy to check that the previous data give a sesqui-functor

$$\Omega^{(n)}\colon n\text{-Gpd}_\star \to n\text{-Gpd}_\star$$

contravariant on lax $n$-transformations.

7.3. PROPOSITION. *There are two strict natural isomorphisms of sesqui-functors*

$$\begin{array}{ccc} & (n\text{-}1)\text{-Gpd}_\star & \\ {\scriptstyle \Pi_1^{(n)}}\nearrow & \simeq & \searrow{\scriptstyle D^{(n)}} \\ n\text{-Gpd}_\star & \xrightarrow[\Omega^{(n)}]{} & n\text{-Gpd}_\star \end{array}$$

$$\begin{array}{ccc} n\text{-Gpd}_\star & \xrightarrow{\Pi_1^{(n)}} & (n\text{-}1)\text{-Gpd}_\star \\ {\scriptstyle \Omega^{(n)}}\searrow & \simeq & \nearrow{\scriptstyle \Pi_0^{(n)}} \\ & n\text{-Gpd}_\star & \end{array}$$

Proof. The second isomorphism follows from the first one composing with $\Pi_0^{(n)}$. As far as the first one is concerned, we recover a natural isomorphism of pointed $n$-groupoids $\theta_{\mathbb{C}} \colon D^{(n)}(\Pi_1^{(n)}\mathbb{C}) \to \Omega^{(n)}\mathbb{C}$ as a special case of that given in Proposition A.1. It suffices to let $\theta_{\mathbb{C}} = \mathfrak{S}_{\mathbb{C}}^{\star;\star}$. ■

We are going to prove the main result of this section: the sesqui-functors $\Omega^{(n)}$ and $\Pi_1^{(n)}$ preserve exact sequences. We need two preliminary lemmas.

7.4. LEMMA. *The sesqui-functor $\Omega^{(n)} \colon n\text{-}Gpd_\star \to n\text{-}Gpd_\star$ preserves h-pullbacks.*

Proof. (Throughout the proof we will omit the superscripts $(n)$.) Let us consider an $h$-pullback

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{Q} & \mathbb{C} \\ {\scriptstyle P}\downarrow & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in $n$-Gpd$_\star$. By the universal property of $h^2$-pullbacks we get $\Omega(\phi)$ as in the diagram

$$\begin{array}{ccc} \Omega(\mathbb{Q}) & \xrightarrow{\Omega(P)} & \Omega(\mathbb{A}) \\ {\scriptstyle \Omega(Q)}\downarrow & \underset{\Omega(\phi)}{\nearrow} & \downarrow{\scriptstyle \Omega(F)} \\ \Omega(\mathbb{C}) & \xrightarrow[\Omega(G)]{} & \Omega(\mathbb{B}) \end{array}$$

Further let us consider the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{M} & \Omega(\mathbb{A}) \\ {\scriptstyle N}\downarrow & \overset{\omega}{\Nearrow} & \downarrow{\scriptstyle \Omega(F)} \\ \Omega(\mathbb{C}) & \xrightarrow[\Omega(G)]{} & \Omega(\mathbb{B}) \end{array}$$

that by Proposition 7.3 can be redrawn

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{M} & D(\mathbb{A}_1(\alpha,\alpha)) \\ {\scriptstyle N}\downarrow & \overset{\omega}{\Nearrow} & \downarrow{\scriptstyle D(F_1^{\alpha,\alpha})} \\ D(\mathbb{C}_1(\star,\star)) & \xrightarrow[D(G_1^{\star,\star})]{} & D(\mathbb{B}_1(\star,\star)) \end{array}$$

Applying $\Pi_0$ one gets

$$\begin{array}{ccc} \Pi_0(\mathbb{X}) & \xrightarrow{\Pi_0(M)} & \mathbb{A}_1(\star,\star) \\ {\scriptstyle \Pi_0(N)}\downarrow & \overset{\Pi_0(\omega)}{\Nearrow} & \downarrow{\scriptstyle F_1^{\star,\star}} \\ \mathbb{C}_1(\star,\star) & \xrightarrow[G_1^{\star,\star}]{} & \mathbb{B}_1(\star,\star) \end{array}$$

Since $\mathbb{Q}_1(\star,\star)$ is defined as an $h$-pullback, its universal property yields a unique morphism $L\colon \Pi_0(\mathbb{X}) \to \mathbb{Q}_1(\star,\star)$ such that

$$L \cdot P_1^{\star,\star} = \Pi_0(M), \quad L \cdot Q_1^{\star,\star} = \Pi_0(N), \quad L \cdot \phi_1^{\star,\star} = \Pi_0(\omega)$$

Using Proposition 5.4 and Proposition 7.3, we obtain from $L$ the required factorization $\mathbb{X} \to D(\mathbb{Q}_1(\star,\star)) \simeq \Omega(\mathbb{Q})$. ■

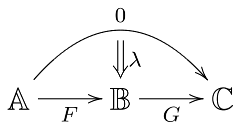
7.5. Lemma. *The sesqui-functor $\Omega^{(n)}\colon n\text{-}Gpd_\star \to n\text{-}Gpd_\star$ preserves $h$-surjective morphisms.*

Proof. This is straightforward. Let $L\colon \mathbb{K} \to \mathbb{A}$ be an $h$-surjective morphism. Then

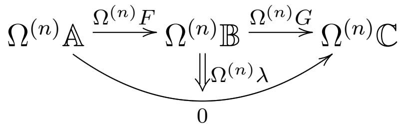
$$\Omega^{(n)}(L) = D^{(n)}(L_1^{\star,\star})$$

Now $L_1^{\star,\star}$ is $h$-surjective by definition since $L$ is, and clearly $D^{(n)}$ preserves $h$-surjective morphisms. ■

7.6. Proposition. *Given an exact sequence in $n$-Gpd$_\star$*




*the sequence*




*is exact in ($n$-1)-Gpd$_\star$.*

Proof. Let $L\colon \mathbb{A} \to \mathbb{K}^*(G)$ be the $h$-surjective comparison with the $h$-kernel of $G$. By Lemma 7.4 above, $\Omega^{(n)}L$ is the comparison with the $h$-kernel of $\Omega^{(n)}G$, and it is $h$-surjective by Lemma 7.5. ■

7.7. Corollary. *The sesqui-functor $\Pi_1^{(n)}\colon$ $n$-Gpd$_\star \to$ ($n$-1)-Gpd$_\star$ preserves exact sequences, reversing the direction of the 2-morphism.*

## 8. The fibration sequence of an $n$-functor

In this section we construct an exact sequence of the form

$$\Omega^{(n)}\mathbb{B} \xrightarrow{\Omega^{(n)}F} \Omega^{(n)}\mathbb{C} \xrightarrow{\nabla} \mathbb{K}^*(F) \xrightarrow{K^*(F)} \mathbb{B} \xrightarrow{F} \mathbb{C}$$

starting from a pointed $n$-functor $F$. We need some lemmas.

8.1. Lemma.

1. *Let $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ be morphisms of $n$-groupoids. If $G$ is an equivalence and $F \cdot G$ is $h$-surjective, then $F$ is $h$-surjective.*

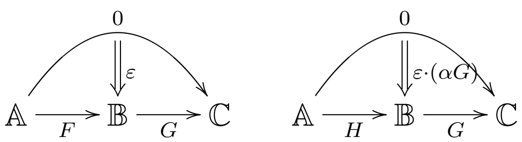
2. *Let $\alpha\colon F \Rightarrow G\colon \mathbb{B} \to \mathbb{C}$ be a 2-morphism of $n$-groupoids. $F$ is $h$-surjective if, and only if, $G$ is $h$-surjective.*

Proof. The proof of the first part, by induction, is strightforward. As far as the second part is concerned, observe that applying the inductive step we have that

$$\mathbb{C}_1(c_0, c_0') \xrightarrow{G_1^{c_0,c_0'}} \mathbb{D}_1(Gc_0, Gc_0') \xrightarrow{\alpha_0 c_0 \circ -} \mathbb{D}_1(Fc_0, Gc_0')$$

is $h$-surjective. Since $\alpha_0 c_0 \circ -$ is an equivalence, we conclude using the first part of the lemma. ∎

8.2. Lemma. *Let $\alpha\colon F \Rightarrow H\colon \mathbb{A} \to \mathbb{B}$ be a 2-morphism of pointed $n$-groupoids. Consider also the following diagrams*



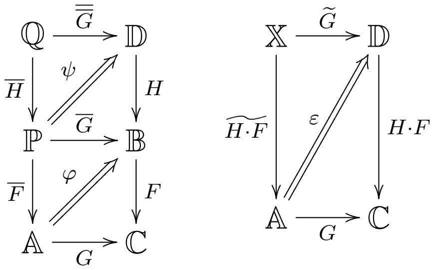

*If $(F, \varepsilon, G)$ is exact, then $(H, \varepsilon \cdot (\alpha G), G)$ is exact.*

Proof. Let $F', H'\colon \mathbb{A} \to \mathbb{K}^*(G)$ be the canonical factorizations of $F$ and $H$ through the $h$-kernel. Using the universal property of the $h^2$-kernel $\mathbb{K}^*(G)$ we get a 2-morphism $\alpha'\colon F' \Rightarrow H'$ and we conclude using the second part of Lemma 8.1. ∎

8.3. Lemma. *Consider*

$$\mathbb{A} \xrightarrow{G} \mathbb{C} \xleftarrow{F} \mathbb{B} \xleftarrow{H} \mathbb{D}$$

*and the following $h$-pullbacks in $n$-Gpd:*



*Consider also the canonical morphisms*

$$\nabla\colon \mathbb{X} \to \mathbb{P} \;\; \textit{such that} \;\; \nabla \cdot \overline{F} = \widetilde{H \cdot F},\ \nabla \cdot \overline{G} = \widetilde{G} \cdot H,\ \nabla \cdot \varphi = \varepsilon$$

$$L\colon \mathbb{X} \to \mathbb{Q} \;\; \textit{such that} \;\; L \cdot \overline{H} = \nabla,\ L \cdot \overline{\overline{G}} = \widetilde{G},\ L \cdot \psi = id$$

*Then $L\colon \mathbb{X} \to \mathbb{Q}$ is an equivalence.*

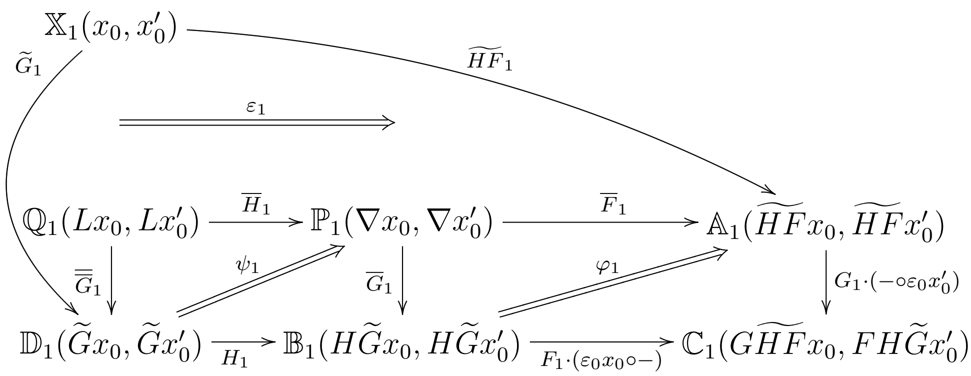
Proof. By induction on $n$.
1) For $n = 1$, the result immediately follows from the fact that in this case $h$-pullbacks are bilimits, so that they are defined up to equivalence.
2) For $n > 1$, let us check that $L$ is essentially surjective. An object in $\mathbb{Q}$ has the form

$$q_0 = ((a_0, c_1 \colon Ga_0 \to Fb_0, b_0), b_1 \colon b_0 \to Hd_0, d_0)$$

whereas an object in $\mathbb{X}$ has the form

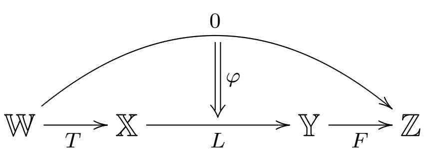
$$x_0 = (a_0, \gamma_1 \colon Ga_0 \to F(Hd_0), d_0)$$

with $L(x_0) = ((a_0, \gamma_1, Hd_0), =, d_0)$. Therefore, for a given object $q_0 \in \mathbb{Q}$, we put $x_0 = (a_0, c_1 \cdot F(b_1), d_0)$ and we have a 1-cell $q_0 \to L(x_0)$ with the identity as components in $\mathbb{A}$ and $\mathbb{D}$, and $b_1$ as component in $\mathbb{B}$.
Finally, to prove that $L_1^{x_0,x'_0} \colon \mathbb{X}_1(x_0, x'_0) \to \mathbb{Q}_1(Lx_0, Lx'_0)$ is an equivalence of ($n$-1)-groupoids we can apply the inductive hypothesis. Indeed, since $L \cdot \psi = id$ and $\nabla \cdot \varphi = \varepsilon$, $L_1^{x_0,x'_0}$ is constructed using $h$-pullbacks precisely as $L$ starting from




■

8.4. Lemma. *Consider the following diagram in $n$-Gpd$_\star$:*



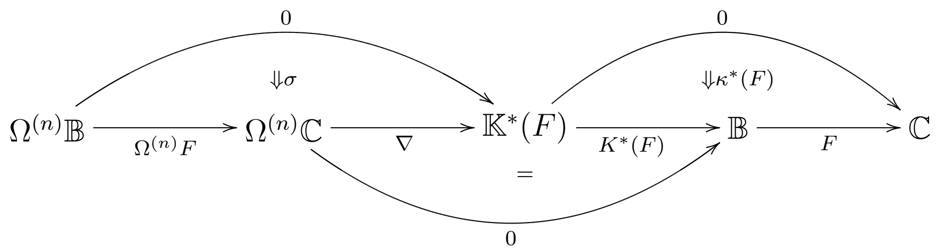

*If $(T \cdot L, \varphi, F)$ is exact and $L$ is an equivalence, then $(T, \varphi, L \cdot F)$ is exact.*

Proof. Let $S\colon \mathbb{K}^*(L \cdot F) \to \mathbb{K}^*(F)$ be the factorization of $(K^*(L \cdot F) \cdot L, \kappa^*(L \cdot F))$ through the $h$-kernel of $F$. Clearly $S$ is the composite of two equivalences (the first one coming from Lemma 8.3 applied to

$$\mathbb{I} \xrightarrow{\ \star\ } \mathbb{Z} \xleftarrow{\ F\ } \mathbb{Y} \xleftarrow{\ L\ } \mathbb{X}$$

and the second one obtained by pulling back $L$ along $K^*(F)$), thus it is itself an equivalence. Consider now the factorization $T'\colon \mathbb{W} \to \mathbb{K}^*(L \cdot F)$ of $(T, \varphi)$ through the $h$-kernel of $L \cdot F$, and the factorization $G\colon \mathbb{W} \to \mathbb{K}^*(F)$ of $(T \cdot L, \varphi)$ thorugh the $h$-kernel of $F$. By uniqueness in the universal property of the $h$-kernel, $T' \cdot S = G$. Since $S$ is an equivalence and, by assumption, $G$ is $h$-surjective, we conclude by Lemma 8.1 that $T'$ is $h$-surjective. ■

8.5. Proposition. *Let $F\colon \mathbb{B} \to \mathbb{C}$ be an $n$-functor in $n$-*Gpd$_\star$*. There is a sequence*




*which is exact in $\Omega^{(n)}\mathbb{C}$, $\mathbb{K}^*(F)$, and $\mathbb{B}$.*

Proof. 1) Exactness in $\mathbb{B}$: obvious.
2) Exactness in $\mathbb{K}^*(F)$: applying Lemma 8.3 to $\mathbb{I} \xrightarrow{\ \star\ } \mathbb{C} \xleftarrow{\ F\ } \mathbb{B} \xleftarrow{\ \star\ } \mathbb{I}$ we get an equivalence

$$L\colon \Omega^{(n)}\mathbb{C} \to \mathbb{K}_*(K^*F) \text{ such that } L \cdot K_*(K^*F) = \nabla,\ L \cdot \kappa_*(K^*F) = id,$$

and the exactness of $(\nabla, id, K^*(F))$.
3) Exactness in $\Omega^{(n)}\mathbb{C}$: applying Lemma 8.3 to

$$\mathbb{I} \xrightarrow{\ \star\ } \mathbb{K}^*(F) \xrightarrow{K^*(F)} \mathbb{B} \xleftarrow{\ \star\ } \mathbb{I}$$

we get an exact sequence

$$\Omega^{(n)}\mathbb{B} \xrightarrow[\Theta]{} \mathbb{K}_*(K^*F) \xrightarrow[K_*(K^*F)]{} \mathbb{K}^*(F) \qquad (0 \colon \Omega^{(n)}\mathbb{B} \to \mathbb{K}^*(F),\ id \Downarrow)$$

where $\Theta$ is the unique morphism such that $\Theta \cdot K_*(K^*F) = 0$ and $\Theta \cdot \kappa_*(K^*F) = \epsilon_{\mathbb{B}}$. By the universal property of the $h^2$-kernel $\mathbb{K}^*(F)$ we get a 2-morphism $\sigma \colon 0 \Rightarrow \Omega^{(n)}F \cdot \nabla$ such that $\sigma \cdot K^*(F) = \epsilon_{\mathbb{B}}$ and $\sigma * \kappa^*(F) = id$. By the universal property of the $h^2$-kernel $\mathbb{K}_*(K^*F)$ we get a 2-morphism $\lambda \colon \Theta \Rightarrow \Omega^{(n)}F \cdot L$ such that $\lambda \cdot K_*(K^*F) = \sigma$ and $\lambda * \kappa_*(K^*F) = id$. Therefore, following Lemma 8.2, we get the sequence

$$\Omega^{(n)}\mathbb{B} \xrightarrow[\Omega^{(n)}F]{} \Omega^{(n)}\mathbb{C} \xrightarrow[L]{} \mathbb{K}_*(K^*F) \xrightarrow[K_*(K^*F)]{} \mathbb{K}^*(F) \qquad (0 \colon \Omega^{(n)}\mathbb{B} \to \mathbb{K}^*(F),\ \lambda \cdot K_*(K^*F) \Downarrow)$$

exact in $\mathbb{K}_*(K^*F)$. Finally, since $L \cdot K_*(K^*F) = \nabla$ and $\lambda \cdot K_*(K^*F) = \sigma$, following Lemma 8.4 we get the exact sequence

$$\Omega^{(n)}\mathbb{B} \xrightarrow[\Omega^{(n)}F]{} \Omega^{(n)}\mathbb{C} \xrightarrow[\nabla]{} \mathbb{K}^*(F) \qquad (0 \colon \Omega^{(n)}\mathbb{B} \to \mathbb{K}^*(F),\ \sigma \Downarrow)$$

■

Since the sesqui-functor $\Omega^{(n)}$ preserves exact sequences, if we apply it to the sequence of Proposition 8.5 we get another exact sequence

$$\Omega^{(n)}\Omega^{(n)}\mathbb{B} \longrightarrow \Omega^{(n)}\Omega^{(n)}\mathbb{C} \longrightarrow \Omega^{(n)}\mathbb{K}^*(F) \longrightarrow \Omega^{(n)}\mathbb{B} \longrightarrow \Omega^{(n)}\mathbb{C}$$

This sequence and the sequence of Proposition 8.5 can be pasted together. Therefore, iterating the process, we obtain a long exact sequence (which trivializes after $n$ applications).

# 9. The Ziqqurath of a pointed $n$-functor

A different perspective is gained by considering the sesqui-functor $\Pi_1$ in place of $\Omega$. In fact in the longer exact sequences obtained at the end of the previous section, repeated applications of $\Omega$ give structures which are discrete in higher dimensional cells. Their exactness can be investigated in lower dimensional settings, *i.e.* after repeated applications of $\Pi_0$. This is a consequence of the following easy to prove

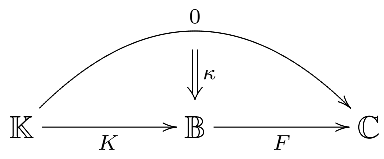
9.1. Lemma. *The sesqui-functor* $\Pi_0$ *commutes with the sesqui-functor* $\Pi_1$, i.e. *for every integer* $n > 1$ *the following diagram is commutative*

$$\begin{array}{ccc} n\text{-Gpd}_\star & \xrightarrow{\Pi_0^{(n)}} & (n\text{-}1)\text{-Gpd}_\star \\ {\scriptstyle \Pi_1^{(n)}}\downarrow & & \downarrow{\scriptstyle \Pi_1^{(n-1)}} \\ (n\text{-}1)\text{-Gpd}_\star & \xrightarrow[\Pi_0^{(n-1)}]{} & (n\text{-}2)\text{-Gpd}_\star \end{array}$$

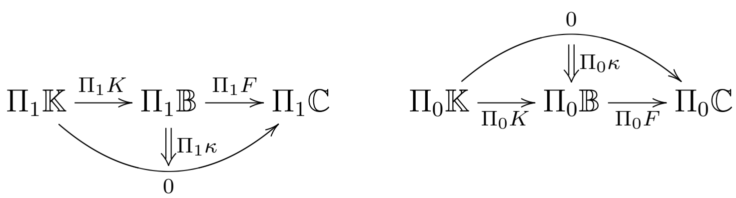
9.2. Remark. In the language of loops, we can restate the above lemma in other terms:

$$\Pi_0(\Pi_0(\Omega(-))) = \Pi_0(\Omega(\Pi_0(-)))$$

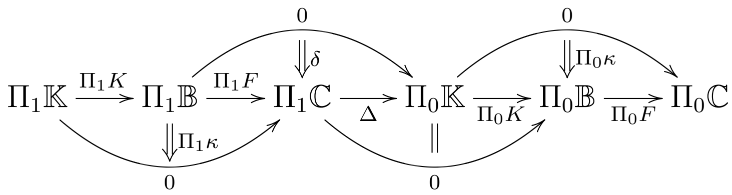
Let now a morphism $F : \mathbb{C} \to \mathbb{D}$ of pointed $n$-groupoids be given. Then the $h$-kernel exact sequence




gives two exact sequences of pointed ($n$-1)-groupoids:

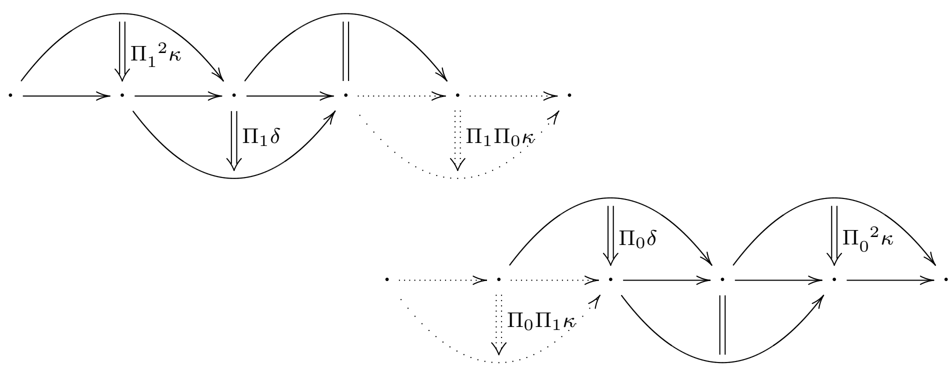


These can be connected together in order to give a six term exact sequence of pointed ($n$-1)-groupoids

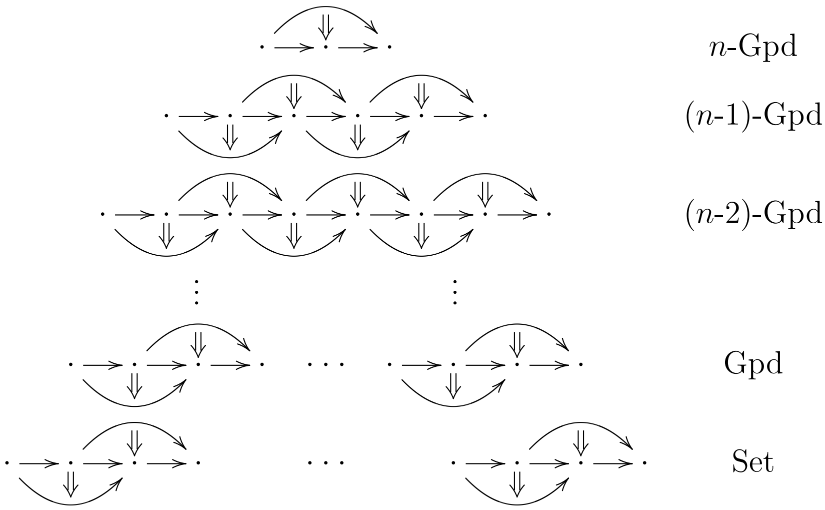



where $\Delta = \Pi_0(\nabla)$ and $\delta = \Pi_0(\sigma)$ (see 8.5 for $\nabla$ and $\sigma$).

Applying $\Pi_0$ and $\Pi_1$, we get two six-term exact sequences. Using the previous lemma, these can be pasted in a nine-term exact sequence of ($n$-2)-groupoids (cells to be pasted are dotted in the diagram):




Iterating the process we obtain a sort of tower, a Ziqqurath, in which the lower is the level, the lower is the dimension and the longer is the length of the sequence.




In particular, the last row counts $3(n{+}1)$ terms. From left to right, there are $3(n\text{-}1)$ abelian groups, 3 groups and 3 pointed sets.

## A. Paths sesqui-functor in $n$-Cat

This section is quite technical. Its aim is to give some explicit constructions that specialize in order to have a good description of the sesqui-functor $\Pi_1^{(n)}$. An observation which can help throughout this section is the analogy between the hom-$(n\text{-}1)$-groupoid of an $n$-groupoid $\mathbb{C}$ and the paths of a topological space. Given an $n$-groupoid ($n$-category) $\mathbb{C}$ and two objects $c_0, c_0'$, we define $\mathbb{P}_{c_0,c_0'}(\mathbb{C})$ by means of the following $h$-pullback:

$$\begin{array}{ccc} \mathbb{P}_{c_0,c_0'}(\mathbb{C}) & \xrightarrow{\;!\;} & \underset{(n)}{\mathbb{I}} \\ {\scriptstyle !}\downarrow & \nearrow\!\!\!\nearrow \;{\scriptstyle \varepsilon_{\mathbb{C}}^{c_0,c_0'}} & \downarrow{\scriptstyle [c_0']} \\ \underset{(n)}{\mathbb{I}} & \xrightarrow[{[c_0]}]{} & \mathbb{C} \end{array} \tag{6}$$

$\mathbb{P}_{c_0,c_0'}$ easily extends to morphisms. In fact for $F : \mathbb{C} \to \mathbb{D}$ one defines

$$\mathbb{P}_{c_0,c_0'}(F) :\; \mathbb{P}_{c_0,c_0'}(\mathbb{C}) \to \mathbb{P}_{c_0,c_0'}(\mathbb{D})$$

by means of the universal property of $h$-pullbacks yielding $\mathbb{P}_{c_0,c_0'}(\mathbb{D})$, for the four-tuple $\langle \mathbb{P}_{c_0,c_0'}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0,c_0'} \cdot F\rangle$. It is easy to see that this makes

$\mathbb{P}_{c_0,c_0'}(-)$ functorial. Unfortunately this does not extend straightforward to 2-morphisms. In fact for a pair of parallel morphisms $F, G : \mathbb{C} \to \mathbb{D}$, $\mathbb{P}_{c_0,c_0'}(F)$ and $\mathbb{P}_{c_0,c_0'}(G)$ are no longer parallel.
Indeed in applying the same argument as for defining $\mathbb{P}_{c_0,c_0'}(-)$ on morphisms, the corresponding diagram suggests to consider the 0-composition of 2-morphisms

$$\varepsilon_{\mathbb{C}}^{c_0,c_0'} * \alpha \colon\ \varepsilon_{\mathbb{C}}^{c_0,c_0'} \setminus \alpha \Longrightarrow \varepsilon_{\mathbb{C}}^{c_0,c_0'}/\alpha\ .$$

Hence we can consider the four-tuples

$$\langle \mathbb{P}_{c_0,c_0'}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0,c_0'} \setminus \alpha \rangle \quad \text{and} \quad \langle \mathbb{P}_{c_0,c_0'}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0,c_0'}/\alpha \rangle$$

together with $id_! : ! \Rightarrow !$ (taken two times) and the 3-morphism $\varepsilon_{\mathbb{C}}^{c_0,c_0'} * \alpha$. Applying the universal property of $h^2$-pullbacks we get a 2-morphism

$$\mathbb{P}_{c_0,c_0'}(\alpha) :\ \mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0,c_0'}(G)) \Rightarrow \mathbb{P}_{c_0,c_0'}(F) \circ \mathbb{P}_{[\alpha_{c_0'}]} :\ \mathbb{P}_{c_0,c_0'}(C) \to \mathbb{P}_{Fc_0,Gc_0'}(\mathbb{D})$$

such that $\mathbb{P}_{c_0,c_0'}(\alpha) * \varepsilon_{\mathbb{D}}^{Fc_0,Gc_0'} = \varepsilon_{\mathbb{C}}^{c_0,c_0'} * \alpha$.
We have denoted by $\mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0,c_0'}(G)$ and $\mathbb{P}_{c_0,c_0'}(F) \circ \mathbb{P}_{[\alpha_{c_0'}]}$ the morphisms obtained by applying the 1-dimensional universal property to $\varepsilon_{\mathbb{C}}^{c_0,c_0'} \setminus \alpha$ and $\varepsilon_{\mathbb{C}}^{c_0,c_0'}/\alpha$ respectively. Therefore the symbol $\circ$ involved should be considered just as a typographical suggestion. (Indeed it can be shown that it is a 0-composition of morphisms, but this would lead us far from the point.)

A.1. Proposition. *For every $n$-category $\mathbb{C}$, and every two objects $c_0, c_0'$ in $\mathbb{C}$, there exists a canonical isomorphism*

$$\mathfrak{S}_{\mathbb{C}}^{c_0,c_0'} : D(\mathbb{C}_1(c_0, c_0')) \to \mathbb{P}_{c_0,c_0'}(\mathbb{C})$$

*In the case of pointed $n$-groupoids, this gives a natural isomorphism with components*

$$\mathfrak{S}_{\mathbb{C}}^{*,*} : D(\Pi_1(\mathbb{C})) \to \Omega(\mathbb{C})$$

*where* $\Omega(\mathbb{C}) = \mathbb{P}_{*,*}(\mathbb{C})$

We start by making explicit the $h$-pullback, but first we need to be more precise on units.

A.2. Remark. Let $\mathbb{C}$ be an $n$-category. For a fixed object $c_0$ of $\mathbb{C}$, let us consider the unit $(n$-1$)$-functor ${}^{\mathbb{C}}u^0(c_0) : \underset{(n-1)}{\mathbb{I}} \to \mathbb{C}_1(c_0, c_0)$, that is a pair

$$\begin{array}{rcl} [{}^{\mathbb{C}}u^0(c_0)]_0 & : & * \mapsto id(c_0) \quad \in [\mathbb{C}_1(c_0, c_0')]_0 \\ [{}^{\mathbb{C}}u^0(c_0)]_1 & : & \underset{(n-2)}{\mathbb{I}} \mapsto [\mathbb{C}_1(c_0, c_0')]_1(id(c_0), id(c_0)) \end{array}$$

Now, by functoriality we get the interchange

$$[{}^{\mathbb{C}}u^0(c_0)]_1 = {}^{\mathbb{C}}u^1(id(c_0)) = {}^{\mathbb{C}_1(c_0,c_0)}u^0(id(c_0))$$

and this allows the following explicit definition:

$${}^{\mathbb{C}}u^0(c_0) = \langle u^{(1)}(c_0), u^{(2)}(c_0), \cdots, u^{(n)}(c_0)\rangle$$

where $u^{(k)}(c_0)$ is the identity $k$-cell over $c_0$.

In the rest of this section the $n$-category $\mathbb{P}_{c_0,c_0'}(\mathbb{C})$ will be denoted by $\mathbb{Q}$.

A.3. Proposition. *Given the h-pullback of $n$-categories*

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\ !\ } & \underset{(n)}{\mathbb{I}} \\ {\scriptstyle !}\downarrow & \nearrow\!\!\!\!\nearrow{\scriptstyle \varepsilon} & \downarrow{\scriptstyle [c_0']} \\ \underset{(n)}{\mathbb{I}} & \xrightarrow[{[c_0]}]{} & \mathbb{C} \end{array} \tag{7}$$

*and $c_k, c_k' \colon c_{k-1} \to c_{k-1}'$, the hom-(n-k)-category*

$$\mathbb{Q}_k\left(\left(u^{(k-1)}(*), c_k, u^{(k-1)}(*)\right), \left(u^{(k-1)}(*), c_k', u^{(k-1)}(*)\right)\right)$$

*is well-defined and it is given by h-pullback over the pair $\langle [c_k], [c_k']\rangle$.*

The proof by induction can be found in [13], and it yields immediately:

A.4. Corollary. *The 2-morphism $\varepsilon$ is given explicitly by*

$$\varepsilon = \langle \varepsilon_0, [\varepsilon_1^{-,-}]_0, \ldots, [\varepsilon_{n-1}^{-,-}]_0, =\rangle$$

*where*

$$\begin{array}{rcl} [\varepsilon_k^{(*,c_{k-1},*),(*,c_{k-1}',*)}]_0 : \mathbb{Q}_k((*, c_{k-1}, *), (*, c_{k-1}', *)) & \to & \mathbb{C}_k(c_{k-1}, c_{k-1}') \\ (*,\ c_{k-1} \xrightarrow{c_k} c_{k-1}',*) & \mapsto & c_k. \end{array}$$

A.5. Corollary. *With notation as above,*

$$D(\Pi_0(\mathbb{Q})) = \mathbb{Q}.$$

In order to describe all compositions of the $n$-category $\mathbb{Q}$, it suffices to study the 0-composition in $\mathbb{Q} = \mathbb{P}_{c_0,c_0'}(\mathbb{C})$, because $k$-compositions (for $k > 0$) are implicit in the inductive definition. *Ditto* for units. This is reported in the lemma below, which together with the following one establishes a link between the globular and the inductive point of view. The interested reader can find the proofs in [13].

A.6. Lemma. *Let $c_1, c_1', c_1'' : c_0 \to c_0'$ be fixed in $\mathbb{C}$. Given*

$$c_k : c_1 = \Rightarrow c_1' \,, \qquad c_k' : c_1 = \Rightarrow c_1'$$

*with $1 < k \leq n$, the following equations hold:*

$$(*, c_k, *) \,{}^{\mathbb{Q}}\circ^0\, (*, c_k, *) = (*, c_k \,{}^{\mathbb{C}}\circ^1\, c_k', *);$$

$$\left[{}^{\mathbb{Q}}u^0((*, c_1, *))\right]_k = \left(*, \left[{}^{\mathbb{C}}u^1(c_1)\right], *\right).$$

A.7. Lemma. *Let $c_0, c_0'$ be objects of an $n$-category $\mathbb{C}$. The assignment*

$$\mathfrak{S}_{\mathbb{C}}^{c_0,c_0'} = \mathfrak{S} : D(\mathbb{C}_1(c_0, c_0')) \to \mathbb{P}_{c_0,c_0'}(\mathbb{C}) = \mathbb{Q}$$

*given explicitly by $\mathfrak{S} = \langle \mathfrak{S}_0, \mathfrak{S}_1, \ldots, \mathfrak{S}_n \rangle$ with $\mathfrak{S}_{i-1}(c_i) = (*, c_i, *)$ for $i = 1, 2, \ldots, n$, and $\mathfrak{S}_n = \mathfrak{S}_{n-1}$, is an isomorphism of discrete $n$-categories.*

Now that we have developed the machinery, we are able to prove the main result of the section.

Proof of Proposition A.1. From the previous lemmas we have the existence of the canonical isomorphism of $n$-categories $\mathfrak{S}_{\mathbb{C}}^{c_0,c_0'}$ for any pair of objects $c_0, c_0'$. Further, for an $n$-functor $F\colon \mathbb{C} \to \mathbb{D}$ we get a $(c_0, c_0')$-indexed family of commutative squares:

$$\begin{array}{ccc} D(\mathbb{C}_1(c_0, c_0')) & \xrightarrow{D(F_1^{c_0,c_0'})} & D(\mathbb{D}_1(Fc_0, Fc_0')) \\ {\scriptstyle \mathfrak{S}_{\mathbb{C}}^{c_0,c_0'}}\downarrow & & \downarrow{\scriptstyle \mathfrak{S}_{\mathbb{D}}^{Fc_0,Fc_0'}} \\ \mathbb{P}_{c_0,c_0'}(\mathbb{C}) & \xrightarrow[\mathbb{P}_{c_0,c_0'}(F)]{} & \mathbb{P}_{Fc_0,Fc_0'}(\mathbb{D}) \end{array} \qquad (8)$$

We prove this by induction.
For $n = 1$ it is just a diagram of discrete categories. It suffices to verify commutativity on objects. To this end, let us choose a $c_1 : c_0 \to c_0'$. Then $\mathfrak{S}_{\mathbb{D}}(DF(c_1)) = \mathfrak{S}_{\mathbb{D}}(Fc_1) = (*, Fc_1, *)$ and $\mathbb{P}F(\mathfrak{S}_{\mathbb{C}}(c_1)) = \mathbb{P}F(*, c_1, *) = (*, Fc_1, *)$.
For $n > 1$, first we have to show that diagram (8) commutes on objects, but this amounts exactly to what we have just shown for $n = 1$. Thus for $c_1, c_1' : c_0 \to c_0'$ we consider homs:

$$\begin{array}{ccc}
[D(\mathbb{C}_1(c_0,c_0'))]_1(c_1,c_1') & \xrightarrow{[D(F_1^{c_0,c_0'})]_1^{c_1,c_1'}} & [D(\mathbb{D}_1(Fc_0,Fc_0'))]_1(Fc_1,Fc_1') \\
\Big\downarrow{\scriptstyle [\mathfrak{S}_{\mathbb{C}}^{c_0,c_0'}]_1^{c_1,c_1'}} & & \Big\downarrow{\scriptstyle [\mathfrak{S}_{\mathbb{D}}^{Fc_0,Fc_0'}]_1^{Fc_1,Fc_1'}} \\
[\mathbb{P}_{c_0,c_0'}(\mathbb{C})]_1((*,c_1,*),(*,c_1',*)) & \xrightarrow[{[\mathbb{P}_{c_0,c_0'}(F)]_1^{\diamond,\diamond}}]{} & [\mathbb{P}_{Fc_0,Fc_0'}(\mathbb{D})]_1((*,Fc_1,*),(*,Fc_1',*))
\end{array}$$

The definition of $D$ and the previous discussion give

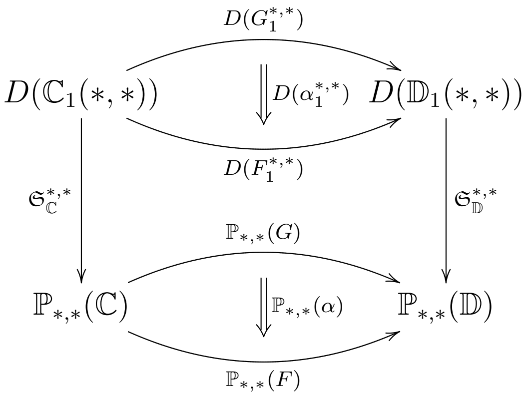
$$\begin{array}{ccc}
D\big([\mathbb{C}_1(c_0,c_0')]_1(c_1,c_1')\big) & \xrightarrow{D\left([F_1^{c_0,c_0'}]_1^{c_1,c_1'}\right)} & D\big([\mathbb{D}_1(c_0,c_0')]_1(Fc_1,Fc_1')\big) \\
{\scriptstyle T_{\mathbb{C}}^{c_1,c_1'}}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{\mathbb{D}}^{Fc_1,Fc_1'}} \\
\mathbb{P}_{c_1,c_1'}\big(\mathbb{C}_1(c_0,c_0')\big) & \xrightarrow[\mathbb{P}_{c_1,c_1'}(F_1^{c_0,c_0'})]{} & \mathbb{P}_{Fc_1,Fc_1'}\big(\mathbb{D}_1(Fc_0,Fc_0')\big)
\end{array}$$

Now, as the $T$'s are just the $\mathfrak{S}$'s given for $n$-1, *i.e.*

$$T_{\mathbb{C}}^{c_1,c_1'} = \mathfrak{S}_{\mathbb{C}_1(c_0,c_0')}^{c_1,c_1'}, \quad T_{\mathbb{D}}^{Fc_1,Fc_1'} = \mathfrak{S}_{\mathbb{D}_1(Fc_0,Fc_0')}^{Fc_1,Fc_1'},$$

the last diagram commutes by induction hypothesis.
All this obviously restricts to $n$-groupoids. Moreover, in the pointed case we obtain a 2-contravariant natural isomorphism of sesqui-functors $\mathfrak{S}\colon \Pi_1 D \Rightarrow \Omega : n\text{-Gpd}_* \to n\text{-Gpd}_*$, *i.e.* a strict natural transformation of sesqui-functors that reverses the direction of 2-morphisms and in

which the assignments on objects are isomorphisms. In fact in $n$-Gpd$_*$ for a 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$, we can express the (strict) naturality condition




$$D(\alpha_1^{*,*}) \cdot \mathfrak{S}_{\mathbb{D}}^{*,*} = \mathfrak{S}_{\mathbb{C}}^{*,*} \cdot \mathbb{P}_{*,*}(\alpha)$$

The proof that this last condition indeed holds is a consequence of a more general (non-pointed) lemma which can be found in [13]. ∎

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